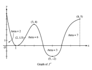
Question 4

The figure above shows the graph of f', the derivative of a twice-differentiable function f, on the closed interval $0 \le x \le 8$. The graph of f' has horizontal tangent lines at $x = 1$, $x = 3$, and $x = 5$. The areas of the regions between the graph of f' and the x-axis are labeled in the figure. The function f is defined for all real numbers and satisfies $f(8) = 4$.



Graph of f'

(a) Find all values of x on the open interval $0 < x < 8$ for which the function f has a local minimum. Justify your answer.
(b) Determine the absolute minimum value of f on the closed interval $0 \le x \le 8$. Justify your answer.
(c) On what open intervals contained in $0 < x < 8$ is the graph of f both concave down and increasing? Explain your reasoning.
(d) The function g is defined by $g(x) = (f(x))^3$. If $f(3) = -\frac{5}{2}$, find the slope of the line tangent to the graph of g at $x = 3$.

(a) At $x=6$, f has a local minimum because $f'(x)$ goes from negative to positive @ $x=6$.

(b) $f(x) = \int f'(x)\,dx$

	Area	Accumulated Area
$0 \to 1$	2	2
$1 \to 4$	6	8
$4 \to 6$	−3	5
$6 \to 8$	7	12

Absolute minimum will occur @ an endpoint or a local minimum. We need to consider $x=0$, $x=6$, & $x=8$.

$f(x) = \int_0^x f'(t)\,dt$

$\int_0^8 f'(t)\,dt = f(8) - \boxed{f(0)}$ ← This allows us to find $f(0)$.

$f(8) = 4$ (Given)

$f(0) = f(8) - \int_0^8 f'(t)\,dt$

$f(0) = 4 - 12$

$f(0) = -8$

Now find $f(6)$.

$\int_6^8 f'(t)\,dt = f(8) - f(6)$

$f(6) = f(8) - \int_6^8 f'(t)\,dt$

$f(6) = 4 - 7$

$f(6) = -3$

Summarizing:

$f(0) = -8 \to -8$ is the absolute minimum on $0 \le x \le 8$.

$f(6) = -3$

$f(8) = 4$

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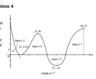


Graph of f'

(a) Find all values of x on the open interval $0 < x < 8$ for which the function f has a local minimum. Justify your answer.
(b) Determine the absolute minimum value of f on the closed interval $0 \le x \le 8$. Justify your answer.
(c) On what open intervals contained in $0 < x < 8$ is the graph of f both concave down and increasing? Explain your reasoning.
(d) The function g is defined by $g(x) = (f(x))^3$. If $f(3) = -\frac{5}{2}$, find the slope of the line tangent to the graph of g at $x = 3$.

(c) f will be concave down when $f'' < 0$ or f' is decreasing.

$0 < x < 1$ & $3 < x < 5$

f will be increasing when $f' > 0$.

f is both concave down & increasing when $0 < x < 1$ & $3 < x < 4$.

(d) $g(x) = [f(x)]^3$

$g'(x) = 3(f(x))^2 \cdot f'(x)$

$g'(3) = 3(f(3))^2 \cdot f'(3)$

$= 3\left(-\frac{5}{2}\right)^2 \cdot 4$ → from graph

$= 3 \cdot \frac{25}{4} \cdot 4$

$= 75$

Question 3

t (minutes)	0	1	2	3	4	5	6
$C(t)$ (ounces)	0	5.3	8.8	11.2	12.8	13.8	14.5

Hot water is dripping through a coffeemaker, filling a large cup with coffee. The amount of coffee in the cup at time t, $0 \le t \le 6$, is given by a differentiable function C, where t is measured in minutes. Selected values of $C(t)$, measured in ounces, are given in the table above.

(a) Use the data in the table to approximate $C'(3.5)$. Show the computations that lead to your answer, and indicate units of measure.

(b) Is there a time t, $2 \le t \le 4$, at which $C'(t) = 2$? Justify your answer.

(c) Use a midpoint sum with three subintervals of equal length indicated by the data in the table to approximate the value of $\frac{1}{6}\int_0^6 C(t)\,dt$. Using correct units, explain the meaning of $\frac{1}{6}\int_0^6 C(t)\,dt$ in the context of the problem.

(d) The amount of coffee in the cup, in ounces, is modeled by $B(t) = 16 - 16e^{-0.4t}$. Using this model, find the rate at which the amount of coffee in the cup is changing when $t = 5$.

(a) $C'(3.5) = \frac{C(4)-C(3)}{4-3} = \frac{12.8-11.2}{1}$ oz/min

(b) MVT

$C(t)$ is continuous & differentiable

$\frac{C(4)-C(2)}{4-2} = \frac{12.8-8.8}{2} = \frac{4}{2} = 2$

By the MVT, there exists a value between 2 & 4 such that $C'(t) = 2$.

t (minutes)	0	1	2	3	4	5	6
$C(t)$ (ounces)	0	5.3	8.8	11.2	12.8	13.8	14.5

$\frac{1}{6}\int_0^6 C(t)\,dt$ → avg. value of $C(t)$

oz·min

$\frac{1}{6}\int_0^6 C(t)\,dt = \frac{2(5.3+11.2+13.8)}{6}$

$\frac{\text{oz}\cdot\text{min}}{\text{min}} = \text{oz}$

The avg amount of coffee in the cup on $0 \le t \le 6$ is $\frac{1}{6} \cdot 2(5.3+11.2+13.8)$ oz.

(d) $B(t) = 16 - 16e^{-0.4t}$

$B'(t) = (-16)(-0.4)e^{-0.4t}$

$\frac{16\cdot 4}{10}$ $\quad B'(t) = 6.4e^{-0.4t}$

6.4 $\quad B'(5) = 6.4e^{(-0.4)(5)} \frac{oz}{min}$

15. The volume of a right circular cone of radius r and height h is given by $V = \frac{1}{3}\pi r^2 h$. Suppose the radius and the height of the cone both increase at a constant rate of $\frac{1}{4}$ meter per minute. At what rate, in cubic meters per minute, is the volume increasing when the height is 4 meters and the radius is 2 meters?

$\frac{dr}{dt}$ → rate of change of radius over time

$\frac{dr}{dt} = \frac{1}{4}\frac{m}{min}$ $\qquad \frac{dh}{dt} = \frac{1}{4}\frac{m}{min}$

$\frac{dV}{dt} = ?$ when $h = 4m$, $r = 2m$

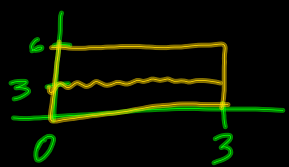
$$V = \boxed{\frac{1}{3}\pi r^2 h}$$

$$\frac{dV}{dt} = \frac{1}{3}\pi r^2 \cdot \frac{dh}{dt} + h \cdot \frac{1}{3}\pi \cdot 2r \cdot \frac{dr}{dt}$$

$$\frac{dV}{dt} = \frac{1}{3}\pi \cdot \cancel{2^2} \cdot \cancel{\frac{1}{4}} + 4 \cdot \frac{1}{3}\pi \cdot \cancel{2 \cdot 2} \cdot \cancel{\frac{1}{4}}$$

$$\frac{dV}{dt} = \frac{1}{3}\pi + \frac{4}{3}\pi = \frac{5}{3}\pi \frac{m^3}{min}$$

28. Let f be a continuous function on the closed interval $[0, 3]$. If $3 \leq f(x) \leq 6$, then the greatest possible value of $\int_0^3 f(x)\,dx$ is



27. The expression

$$\underbrace{\frac{1}{500}}_{\Delta x}\left(\underbrace{\sin\frac{1}{500}}_{f(\Delta x)} + \sin\frac{2}{500} + \sin\frac{3}{500} + \cdots + \sin\frac{500}{500}\right)$$

is a Riemann sum approximation for

$\Delta x = \frac{b-a}{n}$ $\qquad \Delta x \cdot f(\Delta x)$

$$\int_0^1 \sin x\,dx$$

$\frac{d}{dx}\left(\mathbf{e}^x\right) = \mathbf{e}^x$	$\frac{d}{dx}\left(a^x\right) = a^x \ln a$
$\frac{d}{dx}\left(\ln x\right) = \frac{1}{x}$	$\frac{d}{dx}\left(\log_a x\right) = \frac{1}{x \ln a}$

If $\frac{dy}{dx} = \frac{y^2}{2\sqrt{x}}$ and $y = 1$ when $x = 9$, then the solution to the differential equation is

$$\frac{dy}{y^2} = \frac{dx}{2\sqrt{x}}$$

$$\int y^{-2}\,dy = \int \frac{1}{2} \cdot x^{-1/2}\,dx$$

$$\frac{y^{-1}}{-1} = \frac{1}{2} \cdot \frac{x^{1/2}}{1/2} + C$$

$$\frac{-1}{y} = \sqrt{x} + C$$

$$\begin{matrix} x, y \\ (9, 1) \end{matrix}$$

$$\frac{-1}{1} = \sqrt{9} + C$$

$$-1 = 3 + C$$

$$C = -4$$

$$\frac{-1}{y} = \frac{\sqrt{x} - 4}{1}$$

$$y(\sqrt{x} - 4) = -1$$

$$y = \frac{-1}{\sqrt{x} - 4}$$

$$y = \frac{1}{4 - \sqrt{x}}$$