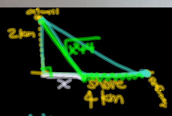
54. ***Minimum Cost*** An offshore oil well is 2 kilometers off the coast. The refinery is 4 kilometers down the coast. If laying pipe in the ocean is twice as expensive as on land, what path should the pipe follow in order to minimize the cost?



Let x = distance down the shore — as measured straight across from oil well

distance in ocean: $\sqrt{x^2+4}$ distance along coast: $4-x$

$\frac{\text{cost}}{\text{km}}$ on land $= c$ (constant)

cost/km in ocean $= 2c$ (also a constant)

$\text{cost} = 2c\sqrt{x^2+4} + c(4-x)$

This is what we want to minimize.

$f(x) = 2c(x^2+4)^{1/2} + 4c - xc$

$f'(x) = 2c \cdot \frac{1}{2}(x^2+4)^{-1/2} \cdot 2x - c$

$f'(x) = 0$ to find critical pts.

$\frac{c \cdot 2x}{\sqrt{x^2+4}} - c = 0$

$\frac{c \cdot 2x}{\sqrt{x^2+4}} = c$

$\frac{2x}{\sqrt{x^2+4}} = 1$

$(2x)^2 = (\sqrt{x^2+4})^2$

$4x^2 = x^2 + 4$

$3x^2 = 4$

$x^2 = \frac{4}{3}$

$x = \pm\frac{2}{\sqrt{3}}$ → x is distance, so only use $\frac{2}{\sqrt{3}}$.

Check to see if this would give a rel. min:

x	$f(x) = 2\sqrt{x^2+4} + 4 - x$
0	$2\sqrt{4} + 4 = 8$
$\frac{2}{\sqrt{3}}$	$2\sqrt{\left(\frac{2}{\sqrt{3}}\right)^2+4} + 4 - \frac{2}{\sqrt{3}} \approx 7.464$
4	$2\sqrt{4^2+4} + 4 - 4 \approx 8.944$

To minimize cost, the pipeline should be built so that it lands approximately 1.1547 km down the coast. $\left(\frac{2}{\sqrt{3}}\text{ km}\right)$

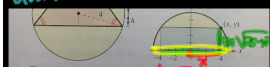


Figure for 26 Figure for 27

27. ***Area*** A rectangle is bounded by the x-axis and the semicircle $y = \sqrt{25 - x^2}$ (see figure). What length and width should the rectangle have so that its area is a maximum?

$A = 2x \cdot \sqrt{25-x^2}$

$A'(x) = 2x\left(\frac{1}{2}(25-x^2)^{-1/2}(-2x)\right) + 2\sqrt{25-x^2}$

$A'(x) = \frac{-2x^2}{\sqrt{25-x^2}} + 2\sqrt{25-x^2}$

$\frac{-2x^2}{\sqrt{25-x^2}} + 2\sqrt{25-x^2} = 0$

$\frac{2x^2}{\sqrt{25-x^2}} = \frac{2\sqrt{25-x^2}}{1}$

$x^2 = 25 - x^2$

$2x^2 = 25$

$x^2 = \frac{25}{2}$

$x = \pm\sqrt{\frac{25}{2}} = \pm\frac{5}{\sqrt{2}}$

x	$f(x) = 2x\sqrt{25-x^2}$
0	0
$5/\sqrt{2}$	$2 \cdot \frac{5}{\sqrt{2}}\sqrt{25 - \frac{25}{2}} = 25$
5	$10\sqrt{0} = 0$

The rectangle has a maximum area when $x = 5/\sqrt{2}$.

$y = \sqrt{25 - \frac{25}{2}} = \frac{5}{\sqrt{2}}$

Area A rectangular page is to contain 30 square inches of print. The margins on each side are 1 inch. Find the dimensions of the page such that the least amount of paper is used.

The twice-differentiable function f is defined for all real numbers and satisfies the following conditions:

$$f(0) = 2, \quad f'(0) = -4, \text{ and } f''(0) = 3.$$

(a) The function g is given by $g(x) = e^{ax} + f(x)$ for all real numbers, where a is a constant. Find $g'(0)$ and $g''(0)$ in terms of a. Show the work that leads to your answers.

(b) The function h is given by $h(x) = \cos(kx) f(x)$ for all real numbers, where k is a constant. Find $h'(x)$ and write an equation for the line tangent to the graph of h at $x = 0$.