7)

x	$f(x)$	$f'(x)$	$g(x)$	$g'(x)$
1	6	4	2	5
2	9	2	3	1
3	10	–4	4	2
4	–1	3	6	7

The functions f and g are differentiable for all real numbers, and g is strictly increasing. The table above gives values of the functions and their first derivatives at selected values of x. The function h is given by $h(x) = f(g(x)) - 6$.

(a) Explain why there must be a value r for $1 < r < 3$ such that $h(r) = -5$.

(b) Explain why there must be a value c for $1 < c < 3$ such that $h'(c) = -5$.

(c) If $g^{-1}(x)$ is the inverse function of g, write an equation for the line tangent to the graph of $y = g^{-1}(x)$ at $x = 2$.

(d) Find the equation of the normal line to $h(x)$ at $x = 1$.

x	$f(x)$	$f'(x)$	$g(x)$	$g'(x)$
1	6	4	2	5
2	9	2	3	1
3	10	–4	4	2
4	–1	3	6	7

$h(x) = f(g(x)) - 6$

$1 < r < 3$

$h(1) = f(g(1)) - 6$
$= f(2) - 6$
$= 9 - 6$
$h(1) = 3$

$h(3) = f(g(3)) - 6$
$= f(4) - 6$
$= -1 - 6$
$h(3) = -7$

By the Intermediate Value Theorem, $h(x)$ must be $= -5$ on $1 < k < 3$ since $h(x)$ is continuous and $h(1) = 3$ & $h(3) = -7$ and -5 is between 3 & -7.

(b) $h(x) = f(g(x)) - 6$

$h'(c) = -5 \rightarrow$ slope of tangent line

Slope of secant: $\dfrac{h(3) - h(1)}{3 - 1} = \dfrac{-7 - 3}{2} = -5$

Since $h(x)$ is continuous & differentiable the MVT applies. ∴ There is at least one value of c on $1 < c < 3$ such that $h'(c) = -5$.

(c) If $g^{-1}(x)$ is the inverse function of g, write an equation for the line tangent to the graph of $y = g^{-1}(x)$ at $x = 2$.

Let $g^{-1}(x) = k(x)$

Inverse functions:

$g(k(x)) = x$

use chain rule

$g'(k(x)) \cdot k'(x) = 1$

$k'(x)$ is the derivative of the inverse.

We need $k'(2)$.

$g'(k(2)) \cdot k'(2) = 1$

$k'(2) = \dfrac{1}{g'(k(2))}$

x	$f(x)$	$f'(x)$	$g(x)$	$g'(x)$
1	6	4	2	5
2	9	2	3	1
3	10	–4	4	2
4	–1	3	6	7

$k(2)$ is

$(1,2)$ is on $g(x)$

$(2,1)$ is on $g^{-1}(x)$ which we have named $k(x)$.

$k(2) = 1$

$k'(2) = \dfrac{1}{g'(1)}$

$k'(2) = \frac{1}{5} \rightarrow$ slope of deriv of inverse of g @ $x=2$

Point: $(2,1)$

$y - 1 = \frac{1}{5}(x - 2)$

(d) Find the equation of the normal line to $h(x)$ at $x = 1$.

Point $(1,3)$ from part a.

$h(x) = f(g(x)) - 6$

$h'(x) = f'(g(x)) \cdot g'(x)$

chain rule

$h'(1) = f'(g(1)) \cdot g'(1)$
$= f'(2) \cdot 5$
$h'(1) = 2 \cdot 5$
$h'(1) = 10$

normal $\rightarrow \perp \rightarrow \frac{-1}{10}$

$y - 3 = -\frac{1}{10}(x - 1)$

5) Find any relative extrema of $y = \arcsin x - x$

Deriv = 0

$$y' = \frac{1}{\sqrt{1-x^2}} - 1$$

$\arcsin 3 = ?$

$\sin\theta = 3$

$$\frac{1}{\sqrt{1-x^2}} = \frac{1}{1}$$

$$\sqrt{1-x^2}^{\,2} = 1^2$$

$$1 - x^2 = 1$$

$$x^2 = 0$$

$$x = 0$$

Test in y' to determine if this is a relative extrema.

$$y' = \frac{1}{\sqrt{1-x^2}} - 1$$

$\frac{1}{\sqrt{1-\frac{1}{4}}} - 1$ (+) at $-\frac{1}{2}$; $\frac{1}{\sqrt{1-\frac{1}{4}}} - 1$ (+) at $\frac{1}{2}$

$-1 \quad -\frac{1}{2} \quad 0 \quad \frac{1}{2} \quad 1$

$$\frac{1}{\sqrt{3/4}} - 1$$

$$\frac{1}{\sqrt{3}} \quad \frac{2}{\sqrt{3}} > 1$$

The derivative does not change signs @ $x=0$ $\therefore$ there are no relative extrema.

II. Derivatives

Concept of the derivative

- Derivative presented graphically, numerically, and analytically.
- Derivative interpreted as an instantaneous rate of change.
- Derivative defined as the limit of the difference quotient.
- Relationship between differentiability and continuity.

Derivative at a point

- Slope of a curve at a point. Examples are emphasized, including points at which there are vertical tangents and points at which there are no tangents.
- Tangent line to a curve at a point and local linear approximation.
- Instantaneous rate of change as the limit of average rate of change.
- Approximate rate of change from graphs and tables of values.

Derivative as a function

- Corresponding characteristics of graphs of f and f'.
- Relationship between the increasing and decreasing behavior of f and the sign of f'.
- The Mean Value Theorem and its geometric interpretation.
- Equations involving derivatives. Verbal descriptions are translated into equations involving derivatives and vice versa.

I. Functions, Graphs, and Limits

Analysis of graphs. With the aid of technology, graphs of functions are often easy to produce. The emphasis is on the interplay between the geometric and analytic information and on the use of calculus both to predict and to explain the observed local and global behavior of a function.

Limits of functions (including one-sided limits)

- An intuitive understanding of the limiting process.
- Calculating limits using algebra.
- Estimating limits from graphs or tables of data.

Asymptotic and unbounded behavior

- Understanding asymptotes in terms of graphical behavior.
- Describing asymptotic behavior in terms of limits involving infinity.
- Comparing relative magnitudes of functions and their rates of change (for example, contrasting exponential growth, polynomial growth, and logarithmic growth).

Continuity as a property of functions

- An intuitive understanding of continuity. (The function values can be made as close as desired by taking sufficiently close values of the domain.)
- Understanding continuity in terms of limits.
- Geometric understanding of graphs of continuous functions (Intermediate Value Theorem and Extreme Value Theorem).

Second derivatives

- Corresponding characteristics of the graphs of f, f', and f''.
- Relationship between the concavity of f and the sign of f''.
- Points of inflection as places where concavity changes.

Applications of derivatives

- Analysis of curves, including the notions of monotonicity and concavity.
- Optimization, both absolute (global) and relative (local) extrema.
- Modeling rates of change, including related rates problems.
- Use of implicit differentiation to find the derivative of an inverse function.
- Interpretation of the derivative as a rate of change in varied applied contexts, including velocity, speed, and acceleration.

Computation of derivatives

- Knowledge of derivatives of basic functions, including power, exponential, logarithmic, trigonometric, and inverse trigonometric functions.
- Derivative rules for sums, products, and quotients of functions.
- Chain rule and implicit differentiation.